GROUP RINGS OF COUNTABLE NON-ABELIAN LOCALLY FREE GROUPS ARE PRIMITIVE

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Abstract

We prove that every group ring of a non-abelian locally free group which is the union of an ascending sequence of free groups is primitive. In particular, every group ring of a countable non-abelian locally free group is primitive. In addition, by making use of the result, we give a necessary and sufficient condition for group rings of ascending HNN extensions of free groups to be primitive, which extends the main result in [16] to the general cardinality case.

1 INTRODUCTION

A ring is (right) primitive if it has a faithful irreducible (right) module. Our purpose in this paper is to study the primitivity of group rings of locally free groups.

A group is called locally free if all of its finitely generated subgroups are free. It is well known that there exist locally free groups which are not free. For example, a properly ascending union of non-abelian free groups of bounded finite rank is infinitely generated and Hopfian (see [14]), and so it is a locally free group which is not free, where a group is Hopfian provided that every surjective endomorphism of that group is an automorphism. An example of uncountable non-free locally free groups can be seen in Higman [10]. It has been seen that a locally free group appears in a subgroup of the fundamental group of a three-dimensional manifold ([8], [1], [11], [15]). The fundamental group of the mapping torus of the standard 2-complex of a free group F with bounding maps the identity and

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an injective endomorphism φ of F, which is called the ascending HNN extension of F corresponding to φ, also has a locally free group as a subgroup.

Recall that the ascending HNN extension F_φ of F corresponding to φ has the presentation $F_\varphi = \langle F, t | t^{-1} f t = \varphi(f) \rangle$. The ascending HNN extension F_φ of a free group F is a well-studied class of groups. For example, F_φ is coherent (Feighn and Handel [6]), where a group is coherent if its finitely generated subgroups are finitely presented. If F is finitely generated then F_φ is Hopfian (Geoghegan, Mihalik, Sapir and Wise [9]). Moreover, Borisov and Sapir [4] have recently shown that it is residually finite. The present author [16] has quite recently shown that the group ring KF_φ is semiprimitive for any field K and it is often primitive provided that F is a non-abelian countable free group. In the proof, it was shown that the group ring of a certain countable locally free group is primitive, and therefore we posed a question: Is it true that the group ring of any locally free group is primitive? In the present paper, we shall give a partial answer to the question:

Theorem 1.1 *Let G be a non-abelian locally free group which has a free subgroup whose cardinality is the same as that of G itself.*

(1) *Let R be a domain (i.e. a ring with no zero divisors). If $|R| \leq |G|$ then the group ring RG is primitive.*

(2) *If K is a field then KG is primitive.*

In particular, every group ring of the union of an ascending sequence of non-abelian free groups over a field is primitive, and so every group ring of a countable non-abelian locally free group over a field is primitive (see Corollary 4.6). Theorem 1.1 corresponds to the result obtained by Formanek [7]. He showed that every group ring RG of a free product G of non-trivial groups (except $G = \mathbb{Z}_2 * \mathbb{Z}_2$) over a domain R is primitive provided that the cardinality of R is not larger than that of G. Motivated by the result, the primitivity of some interesting rings and algebras has been studied (for example, the primitivity of free products of algebras by Lichtman [13], the primitivity of group rings of amalgamated free products by Balogun [2] and the primitivity of semigroup algebras of free products by Chaudhry, Crabb and McGregor [5]). In their papers [2] and [13] (see also [12] and [16]), the method established in [7], which is based on the construction of comaximal ideals, has been applied to obtain the primitivity. This Formanek's method is also available for our study.

In the present paper, we state some graph-theoretic results and apply them to Formanek's method. For the sake of simplicity of the explanation, we consider here the group ring KG over a field K. For a non-zero element u in KG, let $\varepsilon(u)$ be an element in the ideal $KGuKG$ generated by u. Then Formanek's method says that KG is primitive, provided the right ideal ρ generated by the elements $\varepsilon(u)+1$ for all non-zero u in KG is proper. The main difficulty here is how to choose elements $\varepsilon(u)$'s so as to make ρ be proper. That is, the chosen elements $\varepsilon(u)$'s must satisfy that any finite sum of the form $r = \sum(\varepsilon(u)+1)v$ is not the identity element in KG, where v's are elements in KG. In general, $\varepsilon(u)$'s and v's are linear combinations of elements of G; say f's and g's are supports of $\varepsilon(u)$ and v respectively. If r is not the identity element, then it has at least one support of the form fg or g. On the contrary, if r is the identity element, then almost all elements of the form αfg are vanished in r, where α is a non-zero coefficient in K. Then, what can we say about supports f's of $\varepsilon(u)$'s? In order to consider this, regarding the elements of the form fg or g appeared in r as vertices and the equalities of their elements as edges, we use a graph-theoretic method. In section 3, we define an R-graph and an R-cycle, and show that under reasonable conditions, two typical R-graphs have an R-cycle (See Theorem 3.11 and Theorem 3.16). Roughly speaking, if a suitable R-graph for vertices fg's and g's has an R-cycle, then it follows that some supports f_i's of $\varepsilon(u)$'s satisfy the equation $f_1 f_2^{-1} \cdots f_{n-1} f_n^{-1} = 1$. We should then choose $\varepsilon(u)$'s so that their supports f_i's never satisfy such equation.

Now, if φ is an automorphism, that is $\varphi(F) = F$, then the ascending HNN extension F_φ is a cyclic extension of F. On the other hand, If $\varphi(F) \neq F$, then it is a cyclic extension of a locally free group (see Lemma 2.3 (3)). Therefore, by applying Theorem 1.1 to this case, we can establish the primitivity of the group ring KF_φ:

Theorem 1.2 *Let F be a non-abelian free group. Then the following are equivalent:*

(1) KF_φ *is primitive for a field K.*

(2) $|K| \leq |F|$ *or* F_φ *is not virtually the direct product* $F \times \mathbb{Z}$.

(3) $|K| \leq |F|$ *or* $\triangle(G) = 1$, *where* $\triangle(G)$ *is the FC center of* G.

In particular, if F_φ is a strictly ascending HNN extension, that is, $\varphi(F) \neq F$, then KF_φ is primitive for any field K.

This extends the main result [16, Theorem 1.1], which was given for the countable case, to the general cardinality case, and follows the semiprimitivity of KF_φ with any

cardinality (see Corollary 4.7).

2 PRELIMINARIES

Let G be a group and N a subgroup of G. Throughout this paper, we denote by $[G : N]$ the index of N in G. For a group H, G is said to be virtually H if H is isomorphic to N and $[G : N] < \infty$. If g is an element of G, we let $C_N(g)$ denote the centralizer of g in N. Let $C(G)$ be the center of G and $\triangle(G)$ the FC center of G, that is $\triangle(G) = \{g \in G \mid [G : C_G(g)] < \infty\}$. Given a set S, let $|S|$ denote the cardinality of S. If $S \subseteq G$ and $S = \{s_1, \cdots, s_m\}$, $\langle S\rangle = \langle s_1, \cdots, s_m\rangle$ denotes the subgroup of G generated by the elements of S.

The method of Formanek [7] based on the construction of comaximal ideals plays also an important role in our study. We shall give it as follows: Let G be an infinite group, R a ring with identity, and X a set with $|X| = |G|$. Suppose that $|R| \leq |G|$. Let ψ be a bijection from X to the elements of RG except for the zero element. For $x \in X$, let $\varepsilon^*(\psi(x))$ be a non-zero element in the ideal generated by $\psi(x)$ in RG, and let $\rho = \sum_{x\in X} \varepsilon(\psi(x))RG$ be the right ideal of RG, where $\varepsilon(\psi(x)) = \varepsilon^*(\psi(x)) + 1$.

Proposition 2.1 (See [7]) *If ρ is proper then RG is primitive.*

We shall use some basic results on free groups in the last section. For the details, we refer the reader to Lyndon and Schupp [14]. The next assertions on locally free groups are almost obvious. For the sake of completeness, we include a proof.

Lemma 2.2 *Let G be a non-abelian locally free group.*

(1) *The FC center of G is trivial; thus $\Delta(G) = 1$.*

(2) *Let $S = \{v_1, \cdots, v_s, w_1, \cdots, w_t\}$ be a non-empty finite subset of G such that all elements in S are non-trivial, $v_i \neq v_j$ and $w_i \neq w_j$ if $i \neq j$. Then for each $m > 0$, there exist elements $z_1, \cdots, z_m \in G$ which satisfy*

(i) *$v_k z_l w_i z_l = v_h z_n w_j z_n$ if and only if $(k, l, i) = (h, n, j)$,*

(ii) *for $p > 0$, if $\prod_{q=1}^{p}(z_{l_q} w_{i_q} z_{l_q})^{-1}(z_{n_q} w_{j_q} z_{n_q}) = 1$, then either $n_q = l_{(q+1)}$ for some $q \in \{1, \cdots, p-1\}$ or $(l_q, i_q) = (n_q, j_q)$ for some $q \in \{1, \cdots, p\}$.*

Proof. Since G is a non-abelian locally free group, there exists a subset $X \subset G$ with $|X| > 1$ such that $\langle X\rangle$ is freely generated by X in G and $\langle X\rangle \supseteq S$. If $v \in S$ then $C_{\langle X\rangle}(v)$

is cyclic (see [14, Proposition 2.19]), and so $[\langle X\rangle : C_{\langle X\rangle}(v)]$ is not finite, which implies $v \notin \Delta(G)$. Hence we see that $\Delta(G) = 1$ and thus (1) holds.

Now, let $x_1, x_2 \in X$ with $x_1 \neq x_2$, and let n_S be the maximum length of the words in S on X, where the length of a word v is defined for the reduced word equivalent to v on X. We set $n = 2n_S$ and $z_l = x_1^{n+l}x_2x_1^{n+l}$, where $l = 1, 2, \cdots, m$. Then it is easily verified that the above $z_1, \cdots, z_m$ satisfy (i) and (ii). □

Let F be a non-abelian free group, and $F_\varphi = \langle F, t | t^{-1}ft = \varphi(f)\rangle$ the ascending HNN extension of F determined by φ. It is easily verified that every element $g \in F_\varphi$ has a representation of the form $g = t^k f t^{-l}$ where $k, l \geq 0$ and $f \in F$. Combining this with some elementary observations on free groups, we can show the following properties of F_φ:

Lemma 2.3 (See [16, Lemma 2.1, 2.2]) *Let F be a non-abelian free group.*

(1) $\Delta(F_\varphi) = C(F_\varphi)$.

(2) *The following are equivalent:*

(i) $C(F_\varphi) \neq 1$.

(ii) *There exist $n > 0$ and $f \in F$ such that $C(F_\varphi) = \langle t^n f\rangle$.*

(iii) *F_φ is virtually the direct product $F \times \mathbb{Z}$.*

When this is the case, φ is an automorphism of F; thus $\varphi(F) = F$.

(3) *For a non-negative integer i, let F_i be the subgroup of F_φ generated by $\{t^i f t^{-i} \mid f \in F\}$. Then $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_i \subseteq \cdots$ is an ascending chain of free groups, and $F_\infty = \bigcup_{i=1}^{\infty} F_i$ is a normal subgroup of F_φ.*

The next two lemmas are basic results on group rings. We refer the reader to Passman [18] for a more detailed discussion of these questions.

Lemma 2.4 *Let K be a field, G a group and N a subgroup of G.*

(1)(See [20, Theorem 1]) *Suppose that N is normal. If $\triangle(G)$ is trivial and $\triangle(G/N) = G/N$, then KN is primitive implies KG is primitive.*

(2)(See [19, Theorem 3]) *If $\triangle(G)$ is torsion free abelian and $[G : N]$ is finite, then KN is primitive implies KG is primitive.*

Lemma 2.5 (See [17, Theorem 2]) *Let K' be a field and G a group. If $\triangle(G)$ is trivial and $K'G$ is primitive, then for any field extension K of K', KG is primitive.*

Formanek [7] asserts that if G is the direct product of a free group F and the infinite cyclic group $\langle t\rangle$, then KG is primitive if and only if the cardinality of K is not larger than that of F. Combining this with Lemma 2.3 and 2.4 (2), we have

Lemma 2.6 (See [16, Theorem 1.1 (i)]) *Let F be a non-abelian free group, and suppose that $\varphi(F) = F$, that is, $KF_\varphi$is the cyclic extension of F by $\langle t\rangle$. Then the following are equivalent:*

(1) *KF_φ is primitive for a field K.*

(2) *$|K| \le |F|$ or F_φ is not virtually the direct product $F \times \mathbb{Z}$.*

(3) *$|K| \le |F|$ or $\triangle(G) = 1$.*

3 GRAPHICAL INVESTIGATION

Let KG be the group ring of a group G over a field K, and let $a = \sum_{i=1}^{m} \alpha_i f_i$ and $b = \sum_{i=1}^{n} \beta_i g_i$ be in KG ($\alpha_i \neq 0, \beta_i \neq 0$). If $ab = 0$ then for each $f_i g_j$, there exists $f_p g_q$ such that $f_i g_j = f_p g_q$. Suppose that the following k equations hold; $f_1 g_1 = f_2 g_2$, $f_3 g_2 = f_4 g_3, \cdots, f_{2k-3} g_{k-1} = f_{2k-2} g_k$ and $f_{2k-1} g_k = f_{2k} g_1$. Then we can regard the above equations as forming a kind of cycle, and they imply $f_1^{-1} f_2 \cdots f_{2k-1}^{-1} f_{2k} = 1$. That is, the above equations give us a information on supports of a. We can use this idea for a more general case; $a_1 b_1 + \cdots + a_n b_n \in K$ for $a_i, b_i \in KG$ with $a_i = \sum \alpha_{ij} f_{ij}$ and $b_i = \sum \beta_{ik} g_{ik}$. To do this, regarding the elements $f_{ij} g_{ik}$ appeared in $a_i b_i$ as vertices and the equalities of their elements as edges, we use a graph-theoretic method.

In this section, we shall define an R-graph and an R-cycle, and show that under reasonable conditions, two typical R-graphs have an R-cycle. One is called an R-colouring R-graph and another is called an R-simple R-graph which is a special case of an R-colouring. These two results, Theorem 3.11 and Theorem 3.16, are used to prove our main theorem in the next section.

Throughout this section, $\mathcal{G} = (V, E)$ denotes a simple graph; a finite undirected graph which has no multiple edges or loops, where V is the set of vertices and E is the set of edges. For terminology and notations not defined here, we refer to Bondy and Murty [3]. A finite sequence $v_0 e_1 v_1 \cdots e_p v_p$ whose terms are alternately elements e_q's in E and v_q's in V is called a path of length p in $\mathcal{G}$ if $v_{q-1} v_q = e_q \in E$ and $v_q \neq v_{q'}$ for any $q, q' \in \{0, 1, \cdots, p\}$ with $q \neq q'$; simply denoted by $v_0 v_1 \cdots v_p$. Two vertices v and w of $\mathcal{G}$ are said to be connected if there exists a path from v to w in $\mathcal{G}$. Connection is

an equivalence relation on V, and so there exists a decomposition of V into subsets C_i's ($1 \leq i \leq m$) for some $m > 0$ such that $v, w \in V$ are connected if and only if both v and w belong to the same set C_i; each C_i is called a (connected) component of $\mathcal{G}$. Any graph is a disjoint union of components.

Definition 3.1 *Let $\mathcal{G} = (V, E)$ and $\mathcal{G}^* = (V, E^*)$ be simple graphs with the same vertex set V. For $v \in V$, let $U(v)$ be the set consisting of all neighbours of v in $\mathcal{G}^*$ and v itself: $U(v) = \{w \in V \mid vw \in E^*\} \cup \{v\}$. A triple (V, E, E^*) is an R-graph (for a relay-like graph) if it satisfies the following condition* (R)*:*

(R) *If $v \in V$ and C is a component of $\mathcal{G}$, then $|U(v) \cap C| \leq 1$.*

That is, each $U(v)$ has at most one vertex from each component of $\mathcal{G}$. If $\mathcal{G}$ has no isolated vertices, that is, if $v \in V$ then $vw \in E$ for some $w \in V$, then R-graph (V, E, E^*) is called a proper R-graph.

We call $U(v)$ the R-neighbour set of $v \in V$, and set $\mathfrak{U} = \{U(v) \mid v \in V\}$. For $v, w \in V$ with $v \neq w$, it may happen that $U(v) = U(w)$, and so $|\mathfrak{U}| \leq |V|$ generally. If $w, w' \in U(v)$ then the minimum length of paths from w to w' in $\mathcal{G}^*$ is at most 2. Moreover, $|U(v) \cap C| > 1$ for some $v \in V$ and for some component C of $\mathcal{G}$ if and only if there exists a path from w to w' for some $w, w' \in U(v)$ in $\mathcal{G}$. Hence we have

Proposition 3.2 *In the definition 3.1, the condition* (R) *is equivalent to each of the following conditions:*

(R$'$) *If C is a component of $\mathcal{G}$ and C' is a component of $\mathcal{G}^*$ and if there exist $v, w \in C \cap C'$ with $v \neq w$, then the length of any path from v to w in $\mathcal{G}^*$ is longer than* 2.

(R$''$) *If $U \in \mathfrak{U}$ and $v, w \in U$, then there exist no paths from v to w; if $v_0 v_1 \cdots v_m$ is a path in $\mathcal{G}$, then $\{v_0, v_m\} \not\subseteq U$ for any $U \in \mathfrak{U}$.*

By (R$'$), in particular, if $vw \in E^*$ then $vw \notin E$. We say $\mathcal{G} = (V, E)$ to be the base graph of $\mathcal{R} = (V, E, E^*)$. If $E = \emptyset$ then $\mathcal{R}$ is called the empty graph; denoted by $\mathcal{R} = \emptyset$. Clearly, if $\mathcal{R}$ is non-empty then $|\mathfrak{U}| > 1$.

In what follows, let $\mathcal{R} = (V, E, E^*)$ be a non-empty R-graph with $\mathcal{G} = (V, E)$ and $\mathcal{G}^* = (V, E^*)$. For $W \subseteq V$, we define E_W and E^*_W by

$$\begin{aligned} E_W &= \{ww' \mid w, w' \in W \text{ and either } ww' \in E \\ &\qquad \text{or } \ wv_1 \cdots v_m w' \text{ is a path in } \mathcal{G} \ \text{ for some } v_i \in V \setminus W\}, \\ E^*_W &= \{ww' \mid w, w' \in W \text{ and } ww' \in E^*\}. \end{aligned}$$

E_W^* is simply the edges of the subgraph of $\mathcal{G}^*$ generated by W. E_W means that if C is a component of $\mathcal{G}$ with $C \cap W \neq \emptyset$, then $C \cap W$ also becomes a component of $\mathcal{G}_W = (W, E_W)$. It is obvious that $\mathcal{R}_W = (W, E_W, E_W^*)$ is an R-graph. We call $\mathcal{R}_W$ the R-subgraph of $\mathcal{R}$ generated by W, and set $\mathfrak{U}_W = \{U_W(v) = U(v) \cap W \mid v \in W,\ U(v) \in \mathfrak{U}\}$. If E_W coincides with $\{ww' \mid w, w' \in W, ww' \in E\}$, then $\mathcal{R}_W$ is simply called the subgraph of $\mathcal{R}$ generated by W.

The degree $d_W(v)$ of $v \in W$ in $\mathcal{R}_W$ is the number of edges of $\mathcal{G}_W$ incident with v; $d_V(v)$ is simply denoted by $d(v)$. For $X \subseteq W \subseteq V$, $I_W(X)$ denotes $\{x \in X \mid d_W(x) = 0\}$; $I_V(W)$ is simply denoted by $I(W)$. In general, $|I_W(W)| \geq |I(W)|$. In fact, we have

Lemma 3.3 *Let $W \subseteq V$ and $W^c = V \setminus W$. Then*

$$0 \leq |I_W(W)| - |I(W)| \leq |W^c| - |I(W^c)|.$$

The right side equality holds if and only if for each $v \in W^c \setminus I(W^c)$, $d(v) = 1$ and there exists $w \in W$ with $d(w) = 1$ such that $vw \in E$.

Proof. Let $X = I_W(W) \setminus I(W)$ and $Y = W^c \setminus I(W^c)$. Since $0 \leq |X|$ is obvious, it suffices to show that $|X| \leq |Y|$ and the assertion on equality in the statement is true.

Note that $w \in X$ if and only if $d_W(w) = 0$ and $d(w) \neq 0$. Therefore, if $w \in X$ then there exists $v \in Y$ such that $vw \in E$. Then $vw' \notin E$ for any $w' \in X$ with $w' \neq w$. In fact, if $vw' \in E$ then $ww' \in E_W$, which implies a contradiction that $w, w' \notin I_W(W)$. Hence $|X| \leq |Y|$. Since $|X| < \infty$ and $|Y| < \infty$, if $|X| = |Y|$, then for each $w \in X$ (resp. for each $v \in Y$) there exists only one $v \in Y$ (resp. $w \in X$) such that $vw \in E$. From this, if for $w \in X$, $d(w) > 1$ then $ww' \in E$ for some $w' \in W$ with $w \neq w'$, but this implies $w, w' \notin I_W(W)$, a contradiction. Hence $d(w) = 1$ for all $w \in X$. On the other hand, if $d(v) > 1$ for some $v \in Y$ then $vv' \in E$ for some $v' \in Y$ with $v \neq v'$. For these $v, v' \in Y$, as mentioned above, there exists $w, w' \in X$ with $w \neq w'$ such that $vw, v'w' \in E$, which implies a contradiction $w, w' \notin I_W(W)$, because $wvv'w'$ is a path in $\mathcal{R}$ and so $ww' \in E_W$. We have therefore that $d(v) = 1$ for all $v \in Y$.

The converse assertion on equality is obvious. $\Box$

In what follows, for $\pi = v_0 v_1 \cdots v_p$ a path in $\mathcal{G}$, the origin v_0 of π and the terminus v_p of π are denoted by $\mathfrak{o}(\pi)$ and $\mathfrak{t}(\pi)$ respectively.

Definition 3.4 *Let $p > 1$ and let π_q be a path in $\mathcal{G}$ with $v_q = \mathfrak{o}(\pi_q)$ and $w_q = \mathfrak{t}(\pi_q)$ $(1 \leq q \leq p)$. Then a sequence $(\pi_1, \pi_2, \cdots, \pi_p)$ is an R-path of length p in $\mathcal{R}$ if it satisfies the following conditions* (i) *and* (ii)*:*

(i) *All of v_q's and w_q's are different from each other,*

(ii) *$w_q v_{q+1} \in E^*$ for $1 \leq q \leq p-1$.*

If, in addition, it satisfies the following condition (iii)*, then it is an R-cycle of length p in $\mathcal{R}$:*

(iii) *$w_p v_1 \in E^*$.*

In particular, if the length of π_q is 1, that is, $\pi_q = e_q \in E$ for all q, then $(e_1, e_2, \cdots, e_p)$ is called an R-cycle consisting of edges.

It is obvious that $\mathcal{R}$ has an R-cycle if the R-subgraph $\mathcal{R}_W$ has an R-cycle for some $\emptyset \neq W \subseteq V$. A proper R-graph $\mathcal{R}$ is called a clique R-graph, provided that the base graph $\mathcal{G} = (V, E)$ is a clique graph; thus $uv, vw \in E$ implies $uw \in E$. Note that $\mathcal{R}$ is a clique R-graph if and only if every component of $\mathcal{G}$ is a complete graph. Hence, if a clique R-graph has an R-cycle then it also has an R-cycle consisting of edges.

In what follows, $\mathfrak{N}(\mathcal{R}) = \{U \in \mathfrak{U} \mid |U| = 1\}$. We note, if a sequence $(\pi_1, \cdots, \pi_p)$ is an R-cycle in $\mathcal{R}$, then neither $U(\mathfrak{o}(\pi_q))$ nor $U(\mathfrak{t}(\pi_q))$ is in $\mathfrak{N}(\mathcal{R})$ for all $1 \leq q \leq p$.

We consider the following condition (UC) for $\mathfrak{U}$ in $\mathcal{R}$:

(UC) For each U and U' in $\mathfrak{U}$, either $U \cap U' = \emptyset$ or $|U \cap U'| > 1$.

If $\mathfrak{U}$ satisfies (UC) then $\mathfrak{N}(\mathcal{R}) = \emptyset$, because $|U| = |U \cap U| > 1$ for each $U \in \mathfrak{U}$.

Lemma 3.5 *Let $\mathcal{R}$ be a proper R-graph. If $\mathfrak{U}$ satisfies* (UC) *then $\mathcal{R}$ has an R-cycle.*

Proof. Let $v_1 \in V$. Since $\mathcal{R}$ is proper, there exists $w_1 \in V$ such that $e_1 = v_1 w_1 \in E$. Since $|U(w_1)| > 1$, there exists $v_2 \in V$ such that $w_1 v_2 \in E^*$. Then $v_2 v_1 \notin E^*$ by (R). Since $\mathcal{R}$ is proper again, there exists $w_2 \in V$ such that $e_2 = v_2 w_2 \in E$, where $w_2 \neq w_1$ and $w_2 \neq v_1$ because of (R). We see then that (e_1, e_2) satisfies both of (i) and (ii) in Definition 3.4; thus it is an R-path in $\mathcal{R}$. If $U(w_2) \not\subseteq \{v_1, w_1\}$, then we can proceed with this procedure. Since $\mathcal{R}$ is a finite graph, this procedure terminates in a finite number of steps, say, exactly p steps; that is, there exist $p > 1$ and a sequence $\sigma = (e_1, \cdots, e_{p-1})$ of edges with $e_q = v_q w_q$ such that σ is an R-path in $\mathcal{R}$, and in addition, there exists $e_p = v_p w_p \in E$ such that $w_{p-1} v_p \in E^*$ and

$$U(w_p) \subseteq \{v_q, w_q \mid 1 \leq q \leq p-1\}.$$

If there exists $q \in \{1 \leq q \leq p-1\}$ such that $v_q \in U(w_p)$, then the sequence σ contains an R-cycle. In fact, if $w_p = v_q$ for some $q \in \{1, \cdots, p-1\}$ then $q < p-1$ by (R), and then $(\pi_q, e_{q+1}, \cdots, e_{p-1})$ is an R-cycle, where $\pi_q = v_p v_q w_q$. If $v_q \in U(w_p) \setminus \{w_p\}$ for some $q \in \{1, \cdots, p-1\}$; thus $w_p v_q \in E^*$, then $(e_q, \cdots, e_p)$ is an R-cycle. Therefore, we may assume that for each $q \in \{1 \leq q \leq p-1\}$, $v_q \notin U(w_p)$, that is

$$w_p v_q \notin E^* \quad (1 \leq q \leq p-1).$$

Then $w_q \in U(w_p)$ for some $q \in \{1 \leq q \leq p-1\}$. If $w_p = w_q$, $(e_{q+1}, \cdots, e_p)$ is an R-cycle because $w_p v_{q+1} = w_q v_{q+1} \in E^*$. We may assume therefore that $w_p w_q \in E^*$ and q is minimal with this property; thus $q = min\{1 \leq q' \leq p-1 \mid w_p w_{q'} \in E^*\}$. Note that $q < p-1$ by (R). Since $U(w_p) \cap U(v_{q+1}) \supseteq \{w_q\} \neq \emptyset$, we have that $|U(w_p) \cap U(v_{q+1})| > 1$ by the hypothesis (UC). Since $w_p v_{q+1} \notin E^*$, there exists $q' \in \{1 \leq q \leq p-2\}$ with $q' \neq q$ such that $w_p w_{q'} \in E^*$ and $v_{q+1} w_{q'} \in E^*$. By the minimality of q, $q < q'$, in fact, $q+1 < q'$ by (R). Since $w_{q'} v_{q+1} \in E^*$, we see then that $(e_{q+1}, \cdots, e_{q'})$ is an R-cycle. $\square$

In the above lemma, we cannot replace the condition (UC) by $\mathfrak{N}(\mathcal{R}) = \emptyset$. For instance, we have the following example:

Example 3.6 *Let $\mathcal{G} = (V, E)$ be the base graph with $V = \{v_1, \cdots, v_{10}\}$ and $E = \{v_1 v_3, v_2 v_5, v_4 v_7, v_6 v_9, v_8 v_{10}\}$. Let $E^* = \{v_1 v_2, v_3 v_4, v_4 v_5, v_6 v_7, v_7 v_8, v_9 v_{10}\}$. Then we have that $U(v_1) = U(v_2) = \{v_1, v_2\}$, $U(v_3) = \{v_3, v_4\}$, $U(v_5) = \{v_4, v_5\}$, $U(v_4) = \{v_3, v_4, v_5\}$, $U(v_6) = \{v_6, v_7\}$, $U(v_8) = \{v_7, v_8\}$, $U(v_7) = \{v_6, v_7, v_8\}$ and $U(v_9) = U(v_{10}) = \{v_9, v_{10}\}$. In this case, $\mathcal{R} = (V, E, E^*)$ is a non-empty R-graph and $\mathfrak{N}(\mathcal{R}) = \emptyset$ but it has no R-cycles.*

Let $\mathfrak{C}_n(V) = \{V_1, \cdots, V_n\}$ be the set of components of $\mathcal{G}^* = (V, E^*)$. For $1 \leq i \leq n$ and for $v, v' \in V_i$, define $v \simeq v'$ by $U^o(v) = U^o(v')$, where $U^o(v) = U(v) \setminus \{v\}$; thus $U^o(v)$ is the set of neighbours of v in $\mathcal{G}^*$. Clearly, $\simeq$ is an equivalence relation on V_i. If $\mathfrak{c}(V_i) = \{V_{i1}, \cdots, V_{il_i}\}$ is the set of equivalence class of V_i, then V_i is the disjoint union of non-empty V_{ij}'s. We can easily see that

$$\begin{gathered} \text{if } v \in V_i, \text{ there exists } S \subseteq \mathfrak{c}(V_i) \setminus \{V_{ij}\} \text{ such that} \\ U^o(v) = \textstyle\bigcup_{W \in S} W \text{ for all } v \in V_{ij}. \end{gathered} \tag{1}$$

In particular, for each $v, w \in V_{ij}$, $w \notin U^o(v)$. If for each $i \in \{1, \cdots, n\}$ and for each $v, v' \in V_i$, $U(v) \cap U(v') \neq \emptyset$, then $\mathfrak{C}_n(V)$ is called a colouring of $\mathcal{R}$. Note that $\mathfrak{C}_n(V)$ is a colouring if and only if for each i and for each $v, v' \in V_i$, there exists $U \in \mathfrak{U}$ such that

$v, v' \in U$, and so if $\mathfrak{C}_n(V)$ is a colouring then for each i and for each $v, v' \in V_i$, $vv' \notin E$ by (R$''$); thus, in this case, $\mathfrak{C}_n(V)$ is a colouring of the base graph $\mathcal{G}$.

We here consider the following condition (RC) for $\mathfrak{c}(V_i)$ which is stronger than (1) above:

(RC) For each $v \in V_{ij}$, $U^o(v) = V_i \setminus V_{ij}$.

That is, $\mathfrak{c}(V_i)$ satisfies (RC) if and only if $\mathcal{G}^*(V_i, E^*_{V_i})$ is a complete k-partite graph $K_{l_1,\cdots,l_k}$, where $k = |\mathfrak{c}(V_i)|$ and $l_j = |V_{ij}|$.

Let consider the graph $G_i = (\mathfrak{c}(V_i), \mathcal{E})$ with the vertex set $\mathfrak{c}(V_i) = \{V_{i1}, \cdots, V_{il_i}\}$ and the edge set $\mathcal{E} = \{V_{ij}V_{ik} \mid j \neq k, \text{ for } v \in V_{ij}, U^o(v) \supseteq V_{ik}\}$. If $U^o(v) \supseteq V_{ik}$ for $v \in V_{ij}$ then $U^o(v') \supseteq V_{ij}$ for $v' \in V_{ik}$. Combining this with (1), we see that the above definition of $\mathcal{E}$ is well defined. Then, the neighbour set of V_{ij} in G_i is $\mathfrak{c}(V_i) \setminus \{V_{ij}\}$ if and only if for each $v \in V_{ij}$, $U^o(v) = V_i \setminus V_{ij}$ in $\mathcal{R}$, and therefore, $\mathfrak{c}(V_i)$ satisfies (RC) if and only if G_i is isomorphic to the complete graph K_{l_i} on l_i vertices. Now, the definition of $\mathfrak{C}_n(V)$ implies that G_i is a connected graph, and the definition of $\mathfrak{c}(V_i)$ implies that for each distinct vertices V_{ij}, V_{ik} in $\mathfrak{c}(V_i)$, the neighbour set of V_{ij} does not coincide with that of V_{ik}. It easily follows from these above that G_i is isomorphic to the complete graph K_{l_i}, provided $l_i \leq 3$. Hence we have

Remark 3.7 *If* $|\mathfrak{c}(V_i)| \leq 3$, *then* $\mathcal{G}^*_{V_i} = (V_i, E^*_{V_i})$ *is a complete k-partite graph; thus* $\mathfrak{c}(V_i)$ *satisfies* (RC).

In Example 3.6, we can set that $V_{11} = \{v_1\}$, $V_{12} = \{v_2\}$, $V_{21} = \{v_3, v_5\}$, $V_{22} = \{v_4\}$, $V_{31} = \{v_6, v_8\}$, $V_{32} = \{v_7\}$, $V_{41} = \{v_9\}$, $V_{42} = \{v_{10}\}$ and $V_i = \bigcup_j V_{ij}$. Then $\mathcal{G}^*_{V_1} \simeq \mathcal{G}^*_{V_4} \simeq K_{1,1}$ and $\mathcal{G}^*_{V_2} \simeq \mathcal{G}^*_{V_3} \simeq K_{1,2}$.

Let $l_i = |\mathfrak{c}(V_i)|$. If $l_i > 3$, G_i need not be isomorphic to the complete graph K_{l_i} and thus $\mathfrak{c}(V_i)$ need not satisfy (RC) (See Example 3.9 below). Our purpose is to make use of results on R-graphs for proving our main theorem in the next section. From this point of view, it suffices to consider the case when $l_i = 3$. However, in the following consideration, we only need the assumption (RC); we need not to assume the condition $l_i \leq 3$. We therefore define $\mathfrak{C}_n(V)$ to be an R-colouring of $\mathcal{R}$ if for each i, $\mathfrak{c}(V_i)$ satisfies (RC), and investigate when R-colouring R-graphs have an R-cycle. As a result, we can use R-graph theory to analyze more general case than the one of our main theorem (See Corollary 4.5).

Definition 3.8 *Let $\mathcal{R}$ be an R-graph with $\mathfrak{C}_n(V) = \{V_1, \cdots, V_n\}$. If for each $1 \leq i \leq n$, $\mathcal{G}^*_{V_i} = (V_i, E^*_{V_i})$ is a complete k-partite graph; thus $\mathfrak{c}(V_i)$ satisfies* (RC)*, then $\mathfrak{C}_n(V)$ is an R-colouring of $\mathcal{R}$ or $\mathcal{R}$ is an R-colouring R-graph with $\mathfrak{C}_n(V)$.*

If $\mathfrak{C}_n(V)$ is an R-colouring of $\mathcal{R}$, then it is a colouring of $\mathcal{R}$. As has been mentioned in Remark 3.7, in case of $l_i = |\mathfrak{c}(V_i)| \leq 3$, an R-graph is always an R-colouring. However, in case of $l_i > 3$, it is not true. In fact, a colouring need not be an R-colouring. If $\mathfrak{C}_n(V)$ is a colouring of $\mathcal{R}$, then each $V_{ij}, V_{ik} \in \mathfrak{c}(V_i)$, there exists a path π in $G_i = (\mathfrak{c}(V_i), \mathcal{E})$ whose length is 1 or 2 such that the origin $\mathfrak{o}(\pi) = V_{ij}$ and the terminus $\mathfrak{t}(\pi) = V_{ik}$. Hence, for example, if $l_i = 4$, G_i is isomorphic to either the complete graph K_4 or the graph described in the following example:

Example 3.9 *Let $V_i = \{v_1, v_2, v_3, v_4\}$, $U(v_1) = \{v_1, v_2\}$, $U(v_2) = V_i$, $U(v_3) = U(v_4) = \{v_2, v_3, v_4\}$. In this case, we have that $V_{ij} = \{v_j\}$ for $1 \leq j \leq 4$; thus $|\mathfrak{c}(V_i)| = 4$. Then, $v_2 \in U(v_j)$ for all $1 \leq j \leq 4$; thus $U(v_p) \cap U(v_q) \neq \emptyset$ for each $p, q \in \{1, 2, 3, 4\}$, but for $v_1 \in V_{i1}$, $U^o(v_1) = \{v_2\} = V_{i2} \neq V_i \setminus V_{i1}$. Hence, $\mathfrak{c}(V_i)$ satisfies the colouring condition but it fails to satisfy* (RC)*, and certainly, in the graph $G_i = (\mathfrak{c}(V_i), \mathcal{E})$, we see that $\mathcal{E}$ coincides with $\{V_{i1}V_{i2}, V_{i2}V_{i3}, V_{i3}V_{i4}, V_{i4}V_{i2}\}$; thus G_i is not isomorphic to the complete graph K_4.*

Let $\mathcal{R}$ be an R-colouring R-graph with $\mathfrak{C}_n(V)$, where $\mathfrak{C}_n(V) = \{V_1, \cdots, V_n\}$ and $\mathfrak{c}(V_i) = \{V_{i1}, \cdots, V_{il_i}\}$. For $W \subseteq V_i$, We denote by $\mathbf{m}(W)$ the maximum number in $\{|W \cap V_{ij}| \mid 1 \leq j \leq l_i\}$ and by J_W the set $\{j \mid W \cap V_{ij} \neq \emptyset\}$. In general, $\mathbf{m}(V_i) \geq \mathbf{m}(W)$, and clearly, if $\mathbf{m}(W) = 1$ then $\mathcal{G}^*_W = (W, E^*_W)$ is a complete graph, that is, $U_W(v) = W$ for all $v \in W$, and also if $|J_W| > 2$ then $|U_W(v) \cap U_W(w)| > 1$ for all $v, w \in W$. Suppose $|W| > \mathbf{m}(W) + 1$. Then $l_i > 1$ and $|J_W| > 1$. In this case, if $|J_W| = 2$, say $J_W = \{1, 2\}$, and $\mathbf{m}(W) = |W \cap V_{i1}|$, then $\mathbf{m}(W) > 1$ and $|W \cap V_{i2}| = |W| - \mathbf{m}(W) > 1$, which implies that $|U_W(v) \cap U_W(w)| > 1$ for all $v, w \in W$. Hence we have

Remark 3.10 *Let $W \subseteq V_i$.*

(i) *If $\mathbf{m}(W) = 1$ then $U_W(v) = W$ for all $v \in W$.*

(ii) *If $|J_W| > 2$ then $|U_W(v) \cap U_W(w)| > 1$ for all $v, w \in W$.*

(iii) *If $|W| > \mathbf{m}(W) + 1$ then $|U_W(v) \cap U_W(w)| > 1$ for all $v, w \in W$.*

Recall that for $W \subseteq V$, $I(W)$ denotes $\{w \in W \mid d_V(w) = 0\}$.

Theorem 3.11 *Let $n > 1$, and let $\mathcal{R} = (V, E, E^*)$ be an R-colouring R-graph with $\mathfrak{C}_n(V) = \{V_1, \cdots, V_n\}$. Suppose that $|V_i| \geq 2\mathbf{m}(V_i) + 1$ for each $i \in \{1, \cdots, n\}$. If $|I(V)| \leq n$ then $\mathcal{R}$ has an R-cycle.*

Proof. Let $W = V \setminus I(V)$, $W_i = V_i \setminus I(V_i)$ and $m_i = \mathbf{m}(V_i)$ for $i = 1, \cdots, n$.

We prove the statement above by induction on n. First, let $n = 2$. By the assumption,

$$|W_i| = |V_i| - |I(V_i)| \geq 2m_i + 1 - 2 \geq m_i > 0 \quad (i = 1, 2),$$

and so the R-subgraph $\mathcal{R}_W$ (in this case, it is simply the subgraph generated by W) is non-empty; thus $|J_{W_i}| > 0$ for $i = 1, 2$. Moreover,

$$|W| = |V| - |I(V)| \geq |V| - 2 \geq \sum_{i=1}^{2}(2m_i + 1) - 2 = 2(m_1 + m_2). \tag{2}$$

If $|J_{W_i}| = 1$ for $i = 1, 2$, then $|W| \leq m_1 + m_2$, which contradicts (2) above, and so $|J_{W_i}| > 1$ for $i = 1$ or $i = 2$. If $|J_{W_1}| = 1$ then $|J_{W_2}| > 1$ and $|W_1| < |W_2|$ because $|W_1| \leq m_1 < m_1 + 2m_2 \leq |W_2|$ by (2). Since $\mathcal{R}_W$ is proper and $|W_1| < |W_2|$, there exist $v \in W_1$ and $v_1, v_2 \in W_2$ with $v_1 \neq v_2$ such that $vv_1, vv_2 \in E$. However, since $\mathcal{R}$ is an R-colouring R-graph and $|J_{W_2}| > 1$, there exists $U \in \mathfrak{U}_{W_2}$ such that $v_1, v_2 \in U$; this contradicts (R). We see therefore that $|J_{W_i}| > 1$ for both $i = 1$ and $i = 2$, and also that $d(v) = 1$ for all $v \in W$ and $|W_1| = |W_2|$. Again by (2), we have that $|W_i| \geq m_1 + m_2$. In particular, $|W_i| \geq 2$ $(i = 1, 2)$. If $m_1 = 1$ or $m_2 = 1$, say $m_1 = 1$, then $\mathbf{m}(W_1) = 1$ and so $U_W(v) = W_1$ for all $v \in W_1$ by Remark 3.10 (i). For $v_1 \in W_1$, there exists $w_1 \in W_2$ such that $e_1 = v_1w_1 \in E$. Since $|J_{W_2}| > 1$, there exists $v_2 \in W_2$ such that $w_1v_2 \in E^*$, and for this v_2, there exists $w_2 \in W_1$ with $w_2 \neq v_1$ such that $e_2 = v_2w_2 \in E$. Certainly, (e_1, e_2) is an R-cycle because $w_2v_1 \in E^*$. In case of $m_1 > 1$ and $m_2 > 1$, since $|W_i| \geq m_1 + m_2 > \mathbf{m}(W_i) + 1$ for $i = 1, 2$, by virtue of Remark 3.10 (iii), $\mathfrak{U}_W$ satisfies the condition (UC). Hence, by Lemma 3.5, $\mathcal{R}_W$ has an R-cycle and so does $\mathcal{R}$.

Suppose next that $n > 2$ and the statement holds for all numbers between 2 and $n - 1$. If $\mathfrak{U}_W$ satisfies the condition (UC), then it has an R-cycle by Lemma 3.5. We may assume therefore that $\mathfrak{U}_W$ fails to satisfy the condition (UC); thus there exists i such that $|U_{W_i}(v) \cap U_{W_i}(w)| \leq 1$ for some $v, w \in W_i$. By Remark 3.10, for such i, $\mathbf{m}(W_i) = 1$ if and only if $|W_i| = 1$, and it holds that either

$$\begin{array}{rl} & |J_{W_i}| \leq 1 \;\text{ and }\; |W_i| = \mathbf{m}(W_i) \\ \text{or} & |J_{W_i}| = 2 \;\text{ and }\; |W_i| = \mathbf{m}(W_i) + 1. \end{array} \tag{3}$$

In case that there exists $i \in \{1, \cdots, n\}$ such that $|W_i| = \mathbf{m}(W_i)$, say $i = n$, we consider the R-subgraph $\mathcal{R}_{V'}$ with $V' = V \setminus V_n = V_1 \cup \cdots \cup V_{n-1}$. By Lemma 3.3 and the assumption of the statement,

$$\begin{aligned} & |I_{V'}(V')| - |I(V')| \leq |W_n| = \mathbf{m}(W_n), \\ & |I(V')| = |I(V)| - |I(V_n)| \\ \text{and} \quad & |I(V_n)| = |V_n| - |W_n| \geq (2m_n + 1) - \mathbf{m}(W_n), \end{aligned}$$

and so we have that

$$\begin{aligned} |I_{V'}(V')| & \leq |I(V')| + \mathbf{m}(W_n) \\ & = |I(V)| - |I(V_n)| + \mathbf{m}(W_n) \\ & \leq n - (2m_n + 1) + 2\mathbf{m}(W_n) \leq n - 1. \end{aligned}$$

By our induction hypothesis, $\mathcal{R}_{V'}$ has an R-cycle. We may assume therefore that $|W_i| > \mathbf{m}(W_i)$ for all $i \in \{1, \cdots, n\}$. When this is the case, $m(W_i) > 0$ for all i.

Then, by (3), there exists $i \in \{1, \cdots, n\}$ such that $|W_i| = \mathbf{m}(W_i) + 1$ and $|J_{W_i}| = 2$; say $J_{W_i} = \{1, 2\}$. In addition, in this case, as mentioned at (3) above, $\mathbf{m}(W_i) > 1$ because of $|W_i| > 1$. We may here assume $i = n$. Let $W_n = W_{n1} \cup W_{n2}$ with $W_{n1} = \{v_1\}$ and $W_{n2} = \{v_2, \cdots, v_{q+1}\}$, where $q = \mathbf{m}(W_n)$. Since $|V_n| \geq 2m_n + 1 > |V_{n1}| + |V_{n2}|$, there exists $k \in \{1, \cdots, l_n\}$ such that $k \neq 1, 2$ and $I(V_n) \supseteq V_{nk} \neq \emptyset$, where $l_n = |\mathfrak{c}(V_n)|$. We may assume $k = 3$. Let $v_0 \in V_{n3}$ and set $X_n = \{v_0, v_1, v_2\}$ and $V^{(1)} = V_1 \cup \cdots \cup V_{n-1} \cup X_n$. We should note $|V^{(1)}| < |V|$ because $|W_{n2}| = \mathbf{m}(W_n) > 1$. We consider here the R-subgraph $\mathcal{R}_{V^{(1)}}$. It is obvious that $\mathfrak{U}_{X_n}$ satisfies (UC). Let $X_{n1} = \{v_1\}$, $X_{n2} = \{v_2\}$ and $X_{n3} = \{v_0\}$. Then $\mathfrak{c}(X_n) = \{X_{n1}, X_{n2}, X_{n3}\}$ satisfies the R-colouring condition (RC) because $X_{nj} \subseteq V_{nj}$ and $\{V_{n1}, V_{n2}, V_{n3}\}$ satisfies (RC). Hence $\mathfrak{C}_n(V^{(1)}) = \{V_1, \cdots, V_{n-1}, X_n\}$ is an R-colouring of $\mathcal{R}_{V^{(1)}}$. We set $Y_n = V_n \setminus X_n$; thus $Y_n = V \setminus V^{(1)}$. By Lemma 3.3, $|I_{V^{(1)}}(V^{(1)})| \leq |I(V^{(1)})| + |Y_n| - |I(Y_n)|$. Since $|I(V^{(1)})| = |I(V)| - (|I(V_n)| - 1)$ and $|Y_n| - |I(Y_n)| = |W_n| - 2 = \mathbf{m}(W_n) - 1$, we have that

$$\begin{aligned} |I_{V^{(1)}}(V^{(1)})| & \leq |I(V)| - (|I(V_n)| - 1) + (\mathbf{m}(W_n) - 1) \\ & \leq n - |I(V_n)| + \mathbf{m}(W_n). \end{aligned}$$

Moreover, because of $|V_n| \geq 2m_n + 1$, we see that

$$|I(V_n)| = |V_n| - |W_n| \geq (2m_n + 1) - (\mathbf{m}(W_n) + 1) \geq m_n,$$

and hence, $|I_{V^{(1)}}(V^{(1)})| \leq n - m_n + \mathbf{m}(W_n) \leq n$. In addition, $|X_n| \geq 2\mathbf{m}(X_n) + 1$, in fact, $\mathbf{m}(X_n) = 1$ and $|X_n| = 3 = 2\mathbf{m}(X_n) + 1$. That is, $\mathcal{R}_{V^{(1)}}$ satisfy all of the conditions supposed for $\mathcal{R}$ in the statement. Let $W^{(1)} = W_1^{(1)} \cup \cdots \cup W_n^{(1)}$, where $W_i^{(1)} =$

$V_i \setminus I_{V^{(1)}}(V_i)$ $(i = 1, \cdots, n-1)$ and $W_n^{(1)} = \{v_1, v_2\}$. If $\mathfrak{U}_{W^{(1)}}$ fails to the condition (UC) and $|W_i^{(1)}| > \mathbf{m}(W_i^{(1)})$ for all $i \in \{1, \cdots, n\}$, then we can proceed with this procedure, and get R-subgraphs $\mathcal{R}_{V^{(1)}}, \mathcal{R}_{V^{(2)}}, \cdots$. On the other hand, $|V| > |V^{(1)}| > |V^{(2)}| > \cdots$, and therefore, there exists $p > 0$ such that $\mathcal{R}_{V^{(p)}}$ satisfies either (UC) or $|W_i^{(p)}| = \mathbf{m}(W_i^{(p)})$ for some $i \in \{1, \cdots, n\}$. In either case, we have already seen that $\mathcal{R}_{V^{(p)}}$ has an R-cycle. $\square$

In the above theorem, the assumption that $|V_i| \geq 2\mathbf{m}(V_i) + 1$ for each $i \in \{1, \cdots, n\}$ cannot be dropped. Let $\mathcal{R} = (V, E, E^*)$ be the R-graph which is described in Example 3.6, and let $\mathcal{R}' = (V', E', E^{*\prime})$ be the R-graph with $V' = V \cup \{w_1, w_2, w_3, w_4\}$, $E' = E$ and

$$E^{*\prime} = E^* \cup \{v_i w_1, v_j w_2, v_k w_3, v_l w_4 \mid i = 1, 2,\ j = 3, 4, 5,\ k = 6, 7, 8,\ l = 9, 10\}.$$

Then, $V'_{11} = \{v_1\}$, $V'_{12} = \{v_2\}$, $V'_{13} = \{w_1\}$, $V'_{21} = \{v_3, v_5\}$, $V'_{22} = \{v_4\}$, $V'_{23} = \{w_2\}$, $V'_{31} = \{v_6, v_8\}$, $V'_{32} = \{v_7\}$, $V'_{33} = \{w_3\}$, $V'_{41} = \{v_9\}$, $V'_{42} = \{v_{10}\}$, $V'_{43} = \{w_4\}$ and $V'_i = \bigcup_{j=1}^{3} V'_{ij}$. That is, $\mathcal{R}'$ is an R-colouring R-graph with $\mathfrak{C}_4(V') = \{V'_1, V'_2, V'_3, V'_4\}$ and $\mathcal{G}^*_{V'_i} = (V'_i, E^{*\prime}_{V'_i})$ is a complete 3-partite graph. In addition, $I(V') = \{w_1, w_2, w_3, w_4\}$ and so $|I(V')| = 4$. Since $\mathbf{m}(V_1) = \mathbf{m}(V_4) = 1$ and $\mathbf{m}(V_2) = \mathbf{m}(V_3) = 2$, we see that $|V_i| = 2\mathbf{m}(V_i) + 1$ for $i = 1, 4$ but $|V_i| = 2\mathbf{m}(V_i)$ for $i = 2, 3$. As has been pointed out, $\mathcal{R}$ is an R-colouring R-graph with $\mathfrak{C}_4(V)$ and it has no R-cycles. Hence $\mathcal{R}'$ has also no R-cycles, because $V = V' \setminus I(V')$ in $\mathcal{R}'$ and $\mathcal{R}'_V$ is isomorphic to $\mathcal{R}$.

Now, let $\mathcal{R}$ be an R-colouring R-graph with $\mathfrak{C}_n(V) = \{V_1, \cdots, V_n\}$. By Remark 3.10 (i), if $\mathbf{m}(V_i) = 1$ for $i \in \{1, \cdots, n\}$, $U(v) = V_i$ for all $v \in V_i$; thus $\mathcal{G}^*_{V_i} = (V_i, E^*_{V_i})$ is a complete graph. Therefore, if $\mathbf{m}(V_i) = 1$ for every $1 \leq i \leq n$, then $\mathcal{G}^* = (V, E^*)$ is a disjoint union of complete graphs and $\mathfrak{U}$ coincides with the set of components of $\mathcal{G}^*$; thus $\mathfrak{U} = \mathfrak{C}_n(V)$. In such case, we define $\mathfrak{C}_n(V)$ to be a simple R-colouring of $\mathcal{R}$. It is obvious that $\mathfrak{C}_n(V)$ is a simple R-colouring of $\mathcal{R}$ if and only if $\mathcal{R}$ is an R-graph satisfying the condition (SC):

(SC) $\quad U, U' \in \mathfrak{U} \Longrightarrow$ either $U \cap U' = \emptyset$ or $U = U'$.

Definition 3.12 *Let $\mathcal{R} = (V, E, E^*)$ be an R-graph. $\mathcal{R}$ is R-simple if the following conditions are satisfied*:

(i) *$\mathcal{R}$ has a simple R-colouring; thus $\mathcal{R}$ satisfies* (SC),

(ii) *there exist no R-cycles of length* 2 *consisting of edges; if there exists $vw, v'w' \in E$ such that $vv' \in E^*$, then $ww' \notin E^*$.*

Let $\mathcal{R}$ be an R-simple R-graph with the set $\mathfrak{U}$ of R-neighbour sets. By (i), we can define the graph whose vertex set $\mathcal{V} = \mathfrak{U}$ and whose edge set $\mathcal{E} = \{UU' \mid U, U' \in \mathfrak{U}$, there exist $v \in U$ and $v' \in U'$ such that $vv' \in E\}$; the graph $(\mathcal{V}, \mathcal{E})$ is denoted by $\mathcal{R}/\mathfrak{U}$. Then $|\mathcal{V}| = |\mathfrak{U}|$, and for $U \in \mathcal{V}$, $d_{\mathcal{V}}(U) = \sum_{v \in U} d_V(v)$. Moreover, $\mathcal{R}/\mathfrak{U}$ has no loops by (R), and has no multiple edges by (ii); that is, $\mathcal{R}/\mathfrak{U}$ is a simple graph. We call $\mathcal{R}/\mathfrak{U}$ the induced simple graph of $\mathcal{R}$.

If $\mathcal{R}$ has an R-cycle then it induces the cycle in $\mathcal{R}/\mathfrak{U}$. Conversely, if $\mathcal{R}/\mathfrak{U}$ has a cycle then the origin of it in $\mathcal{R}$ is either an R-cycle or a cycle in the base graph $\mathcal{G} = (V, E)$. Hence we have

Lemma 3.13 *Let $\mathcal{R}$ be an R-simple R-graph with the base graph $\mathcal{G} = (V, E)$ and the set $\mathfrak{U}$ of R-neighbour sets. Suppose that $\mathcal{G}$ has no cycles. Then $\mathcal{R}$ has an R-cycle if and only if $\mathcal{R}/\mathfrak{U}$ has a cycle.*

Definition 3.14 *Let $\mathcal{R}$ be an R-simple R-graph with the set $\mathfrak{U}$ of R-neighbour sets. Then U and U' in $\mathfrak{U}$ are said to be R-connected if there exists a finite sequence $U_0 e_1 U_1 \cdots e_p U_p$ whose terms are alternately R-neighbour sets U_q's and edges e_q's in E with $e_q = v_q w_q$ such that $U_0 = U$, $U_p = U'$, $v_q \in U_{q-1}$, $w_q \in U_q$.*

In an R-simple R-graph, 'R-connected' means simply 'connected' in the induced simple graph of it. Since R-connection is an equivalence relation on $\mathfrak{U}$, there exists a decomposition of $\mathfrak{U}$ into subsets $\mathfrak{U}_i$'s $(1 \le i \le m)$ for some $m > 0$ such that $U, U' \in \mathfrak{U}$ are R-connected if and only if both U and U' belong to the same set $\mathfrak{U}_i$. The subgraphs $\mathcal{R}_{W_1}, \cdots, \mathcal{R}_{W_m}$ of $\mathcal{R}$ generated by W_i's are called the R-components of $\mathcal{R}$, provided $W_i = \bigcup_{U \in \mathfrak{U}_i} U$ for each $i \in \{1, \cdots, m\}$ and $V = \bigcup_{i=1}^m W_i$. If $\mathcal{R}$ has exactly one R-component then $\mathcal{R}$ is R-connected.

Recall that $\mathfrak{N}(\mathcal{R}) = \{U \in \mathfrak{U} \mid |U| = 1\}$. We set $\mathfrak{L}(\mathcal{R}) = \{U \in \mathfrak{U} \mid |U| > 2\}$ and $\mathfrak{M}(\mathcal{R}) = \{U \in \mathfrak{U} \mid |U| = 2\}$.

Lemma 3.15 *Let $\mathcal{R} = (V, E, E^*)$ be an R-simple R-graph with the set $\mathfrak{U}$ of R-neighbour sets. Suppose that $\mathcal{R}$ is R-connected. Then $\mathcal{R}$ has an R-cycle if and only if $|V| - |\mathfrak{U}| - \omega + 1 \neq 0$, where ω is the number of components of $\mathcal{G}$.*

In particular, if $\mathcal{R}$ is proper and $|\mathfrak{L}(\mathcal{R})| \ge |\mathfrak{N}(\mathcal{R})|$ then $\mathcal{R}$ has an R-cycle.

Proof. Let $\mathcal{G}_i = (V_i, E_i)$ $(i = 1, \cdots, \omega)$ be the components of the base graph $\mathcal{G} = (V, E)$. Since $\mathcal{G}_i$ is connected, there exists a spanning tree $\mathcal{G}'_i = (V_i, E'_i)$ of $\mathcal{G}_i$. Let $E' = \bigcup_{i=1}^{\omega} E'_i$,

$\mathcal{G}' = (V, E')$ and $\mathcal{R}' = (V, E', E^*)$. It is obvious that $\mathcal{R}'$ is R-simple and R-connected. Moreover, $\mathcal{R}$ has an R-cycle if and only if so does $\mathcal{R}'$. Since $\mathcal{G}'$ has no cycles, by virtue of Lemma 3.13, $\mathcal{R}'$ has an R-cycle if and only if the induced simple graph $\mathcal{R}'/\mathfrak{U} = (\mathcal{V}', \mathcal{E}')$ has a cycle.

On the other hand, since $\mathcal{R}'$ is R-connected, $\mathcal{R}'/\mathfrak{U}$ is connected, and so $\mathcal{R}'/\mathfrak{U}$ is a tree if and only if

$$|\mathcal{E}'| = |\mathcal{V}'| - 1. \tag{4}$$

We set $D = \sum_{U \in \mathcal{V}'} d_{\mathcal{V}'}(U)$ and $d_i = \sum_{v \in V_i} d_V(v)$, where $d_V(v)$ means the degree of v not in $\mathcal{G}$ but in $\mathcal{G}'$. Recall that $d_i = 2|V_i| - 2$ because $\mathcal{G}'_i$ is a tree. Hence we have

$$D = \sum_{i=1}^{\omega} d_i = \sum_{i=1}^{\omega} (2|V_i| - 2) = 2|V| - 2\omega.$$

Since $|\mathcal{V}'| = |\mathfrak{U}|$ and generally $D = 2|\mathcal{E}'|$, the condition (4) can be replaced by $|V| - |\mathfrak{U}| - \omega + 1 = 0$. That is, $\mathcal{R}'/\mathfrak{U}$ has a cycle if and only if $|V| - |\mathfrak{U}| - \omega + 1 \neq 0$.

Now, if $\mathcal{R}$ is proper, $2|V| - 2\omega = D \geq 3|\mathfrak{L}(\mathcal{R})| + 2|\mathfrak{M}(\mathcal{R})| + |\mathfrak{N}(\mathcal{R})|$ and $|\mathfrak{U}| = |\mathfrak{L}(\mathcal{R})| + |\mathfrak{M}(\mathcal{R})| + |\mathfrak{N}(\mathcal{R})|$. Hence, in particular, if $|\mathfrak{L}(\mathcal{R})| \geq |\mathfrak{N}(\mathcal{R})|$, then

$$|V| - |\mathfrak{U}| - \omega + 1 \geq \frac{1}{2}(|\mathfrak{L}(\mathcal{R})| - |\mathfrak{N}(\mathcal{R})|) + 1 > 0,$$

and so $\mathcal{R}$ has an R-cycle. $\Box$

Theorem 3.16 *Let $\mathcal{R} = (V, E, E^*)$ be an R-simple R-graph with the set $\mathfrak{U}$ of R-neighbour sets. Then $\mathcal{R}$ has an R-cycle if and only if there exists an R-component $\mathcal{R}_W = (W, E_W, E^*_W)$ of $\mathcal{R}$ with the set $\mathfrak{U}_W$ of R-neighbour sets such that $|W| - |\mathfrak{U}_W| - \omega + 1 \neq 0$, where ω is the number of components of $\mathcal{G}_W = (W, E_W)$.*

In particular, if $\mathcal{R}$ is proper and $|\mathfrak{L}(\mathcal{R})| \geq |\mathfrak{N}(\mathcal{R})|$ then $\mathcal{R}$ has an R-cycle.

Proof. If there exists an R-cycle in $\mathcal{R}$, then it exists in an R-component. Hence $\mathcal{R}$ has an R-cycle if and only if there exists an R-component $\mathcal{R}_W$ of $\mathcal{R}$ which has an R-cycle. Let $\mathcal{R}_W = (W, E_W, E^*_W)$ be an R-component of $\mathcal{R}$ with the set $\mathfrak{U}_W$ of R-neighbour sets, and let ω be the number of components of $\mathcal{G}_W = (W, E_W)$. By Lemma 3.15, $\mathcal{R}_W$ has an R-cycle if and only if $|W| - |\mathfrak{U}_W| - \omega + 1 \neq 0$.

Now, if $|\mathfrak{L}(\mathcal{R})| \geq |\mathfrak{N}(\mathcal{R})|$ then there exists an R-component $\mathcal{R}_W$ such that $|\mathfrak{L}(\mathcal{R}_W)| \geq |\mathfrak{N}(\mathcal{R}_W)|$, and in addition, if $\mathcal{R}$ is proper then so is $\mathcal{R}_W$. Hence, in this case, $\mathcal{R}_W$ has an R-cycle by Lemma 3.15. This completes the proof. $\Box$

4 PROOF OF THEOREMS

In what follows, let G be a non-abelian locally free group in which there exists a free subgroup H with basis X such that $|H| = |G|$. Since G is non-abelian, we may assume that H is non-abelian. If the rank of H is infinite then $|X| = |H|$. On the other hand, if the rank of H is finite then there exists a free subgroup of H whose rank is countable; for instance, the derived subgroup $[H, H]$ of H is a free group of countable rank. Therefore, we may assume here that $|X| = |G|$. Let R ($\ni 1$) be a ring with no zero divisors. We suppose that $|R| \le |G|$. Since $|X| = |G| \ge \aleph_0$, we can divide X into three subsets X_1, X_2 and X_3 each of whose cardinality is $|X|$. Let σ_i be a bijection from X to X_i ($i = 1, 2, 3$). For $x \in X$, $x^{(i)}$ denote the image of x by σ_i. Let RG be the group ring of G over R. Since $|RG| = |X|$, there exists a bijection ψ from X to $RG \setminus \{0\}$. Let $x \in X$, and let

$$\psi(x) = \sum_{i=1}^{m_x} \alpha_{xi} f_{xi}, \quad \text{where} \quad \alpha_{xi} \in R,\ f_{xi} \in G,\ m_x > 0, \tag{5}$$

each of them depends on x, and $f_{xi} \neq f_{xj}$ if $i \neq j$. Since G is locally free, for the subset $\{x^{(1)}, x^{(2)}, x^{(3)}, f_{xi} \mid 1 \le i \le m_x\}$ of G, there exist elements z_{x1}, z_{x2}, z_{x3} in G which satisfy the assertions of Lemma 2.2 (2). We define $\varepsilon(x)$ by

$$\begin{aligned} \varepsilon(x) &= \sum_{k=1}^{3}\sum_{l=1}^{3} x^{(k)} z_{xl} \psi(x) z_{xl} + 1 \\ &= \sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{i=1}^{m_x} \alpha_{xi} x^{(k)} z_{xl} f_{xi} z_{xl} + 1. \end{aligned} \tag{6}$$

Let $\xi_x(k, l, i) = x^{(k)} z_{xl} f_{xi} z_{xl}$. Then the assertions of Lemma 2.2 (2) mean the the following:

Remark 4.1 (i) $\xi_x(k, l, i) = \xi_x(h, n, j)$ *if and only if* $(k, l, i) = (h, n, j)$.

(ii) *Let* $p > 0$, $1 \le l_q, n_q \le 3$ *and* $1 \le i_q, j_q \le m_x$, *where* $1 \le q \le p$. *If* $\prod_{q=1}^{p} \xi_x(1, l_q, i_q)^{-1} \xi_x(1, n_q, j_q) = 1$, *then either* $n_q = l_{q+1}$ *for some* $q \in \{1, \cdots, p-1\}$ *or* $(l_q, i_q) = (n_q, j_q)$ *for some* $q \in \{1, \cdots, p\}$.

Let ρ be the right ideal of RG generated by all $\varepsilon(x)$'s, that is,

$$\rho = \sum_{x \in X} \varepsilon(x) RG. \tag{7}$$

Let $r = \sum_{t=1}^{m} r_t$ be a non-zero element of ρ, where $0 \neq r_t \in \varepsilon(x_t) RG$ with $x_t \in X$. Then there exist $n_t > 0$, $\beta_{tj} \in R$ with $\beta_{tj} \neq 0$ and $g_{tj} \in G$ such that $r_t = \varepsilon(x_t) \sum_{j=1}^{n_t} \beta_{tj} g_{tj}$ with

$g_{tj} \neq g_{ti}$ $(j \neq i)$. In what follows, we simply write m_t, α_{ti}, x_{tk}, z_{tl} and f_{ti} instead of m_{x_t}, $\alpha_{x_t i}$, $x_t^{(k)}$, $z_{x_t l}$ and $f_{x_t i}$, respectively. By the expression of $\varepsilon(x_t)$ as in (6), we have that

$$r_t = \sum_{k,l=1}^{3} \sum_{i=1}^{m_t} \sum_{j=1}^{n_t} \alpha_{ti}\beta_{tj} x_{tk} z_{tl} f_{ti} z_{tl} g_{tj} + \sum_{j=1}^{n_t} \beta_{tj} g_{tj}, \text{ where } m_t, n_t > 0. \tag{8}$$

To prove Theorem 1.1, by virtue of Proposition 2.1, we shall show that ρ is a proper right ideal of RG. By making use of graph-theoretic results obtained in the previous section, we shall prove $r = \sum_{t=1}^{m} r_t \neq 1$. To connect the problem with our graphical method, we prepare the following notations.

For $1 \leq t \leq m$, let n_t and m_t be as described in (8). we set

$$\begin{aligned} & P_t = \{(t,k,l,i,j) \mid 1 \leq k, l \leq 3,\ 1 \leq i \leq m_t,\ 1 \leq j \leq n_t\} \\ \text{and} \quad & Q_t = \{(t,j) \mid 1 \leq j \leq n_t\}. \end{aligned} \tag{9}$$

Moreover, for $v = (t,k,l,i,j) \in P_t$ and $w = (t,j) \in Q_t$, we set

$$\eta(v) = x_{tk} z_{tl} f_{ti} z_{tl} g_{tj} \ \text{ and } \ \eta(w) = g_{tj}. \tag{10}$$

Then we can replace the expression (8) of r_t by the following expression:

$$r_t = \sum_{v \in P_t} \gamma_v \eta(v) + \sum_{w \in Q_t} \gamma_w \eta(w), \text{ where } \gamma_v = \alpha_{ti}\beta_{tj} \text{ and } \gamma_w = \beta_{tj}. \tag{11}$$

Let $P = \bigcup_{t=1}^{m} P_t$ and $Q = \bigcup_{t=1}^{m} Q_t$. We regard $W = P \cup Q$ as the set of vertices and $E = \{vw \mid v, w \in W, v \neq w \text{ and } \eta(v) = \eta(w)\}$ as the set of edges, and consider the R-graph $\mathcal{R} = (W, E, E^*)$, where $vv' \in E^*$ if and only if $v, v' \in P_t$ such that $v = (t,k,l,i,j)$ and $v' = (t,k',l,i,j)$ with $k \neq k'$; thus $U(v) = \{(t,k',l,i,j) \mid 1 \leq k' \leq 3\}$ for $v = (t,k,l,i,j) \in P_t$, $U(w) = \{w\}$ for $w \in Q_t$ and $\mathfrak{U} = \{U(v) \mid v \in W\}$ (in the proof of Theorem 1.1, in fact, the above vertices set $W = P \cup Q$ is replaced by $V = P^* \cup Q^*$; see below for detail). We shall then show that there exist some isolated vertices in $\mathcal{R}$ which make $r = 1$ false. To do this, by making use of Theorem 3.11, we shall first show that there exists a suitable number of isolated vertices in the subgraph of $\mathcal{G} = (W, E)$ generated by P_{t1} in Lemma 4.4 after preparing two remarks, where $P_{t1} = \{v \mid v = (t,1,l,i,j) \in P_t\}$.

Let $M_t = \{(l,i,j) \mid 1 \leq l \leq 3,\ 1 \leq i \leq m_t,\ 1 \leq j \leq n_t\}$. For $v = (t,k,l,i,j) \in P_t$ and $\mu = (l,i,j) \in M_t$, we write $v = (t,k,\mu)$. Let $\mu = (l,i,j)$ and $\mu' = (l',i',j')$ be in M_t. If $j = j'$ then $\eta(t,k,\mu) = \eta(t,k',\mu')$ if and only if $(k,l,i) = (k',l',i')$ by Remark 4.1 (i). If $(l,i) = (l',i')$, then $\eta(t,1,\mu) = \eta(t,1,\mu')$ implies $g_{tj} = g_{tj'}$, and so $j = j'$; thus $\mu = \mu'$. Therefore, if $\mu \neq \mu'$, then $\eta(t,1,\mu) = \eta(t,1,\mu')$ implies $j \neq j'$ and $(l,i) \neq (l',i')$. Moreover, it is obvious that $\eta(t,1,\mu) = \eta(t,1,\mu')$ holds if and only if $\eta(t,k,\mu) = \eta(t,k,\mu')$ holds for all $k \in \{1,2,3\}$. Hence we have

Remark 4.2 *Let $t \in \{1, \cdots, m\}$, and let $v, v' \in P_t$ with $v = (t, k, \mu)$ and $v' = (t, k', \mu')$, where $\mu = (l, i, j)$ and $\mu' = (l', i', j')$.*

(i) *Suppose that $\eta(v) = \eta(v')$. If $j = j'$, then $v = v'$.*

(ii) *Suppose that $\eta(v) = \eta(v')$. If $k = k'$ and either $j = j'$ or $(l, i) = (l', i')$, then $\mu = \mu'$.*

(iii) *$\eta(t, k, \mu) = \eta(t, k, \mu')$ holds for some $k \in \{1, 2, 3\}$ if and only if it holds for any $k \in \{1, 2, 3\}$.*

For μ and μ' in M_t, define the relation $\mu \sim \mu'$ by $\eta(t, 1, \mu) = \eta(t, 1, \mu')$. It is obvious that $\sim$ is a equivalence relation on M_t. Let $C_t(\mu)$ be the equivalence class of $\mu \in M_t$ and let $N_t = \{\mu \in M_t \mid C_t(\mu) = \{\mu\}\}$. Since $N_t \subseteq M_t$ and $\nu \not\sim \nu'$ for $\nu, \nu' \in N_t$ with $\nu \neq \nu'$, as a complete set of representatives for $M_t/\sim$, we can choose a set T_t which satisfies $N_t \subseteq T_t$. For $\mu = (l, i, j) \in T_t$, let $\gamma_{(t,1,\mu)} = \alpha_{ti}\beta_{tj}$, where $\alpha_{ti}\beta_{tj}$ is as described in (11). We set

$$\gamma^*_{(t,1,\mu)} = \sum_{\mu' \in C_t(\mu)} \gamma_{(t,1,\mu')} \quad \text{and} \quad M^*_t = \{\mu \in T_t \mid \gamma^*_{(t,1,\mu)} \neq 0\}.$$

By the definition of M^*_t and Remark 4.2 (i) (iii), we have

Remark 4.3 *Let $t \in \{1, \cdots m\}$ and $k, k' \in \{1, 2, 3\}$. For $\mu, \mu' \in M^*_t$, suppose $\eta(t, k, \mu) = \eta(t, k', \mu')$. Then $k = k'$ if and only if $\mu = \mu'$.*

Now in (11), replacing P_t by $P^*_t = \{(t, k, \mu) \mid \mu \in M^*_t,\ 1 \leq k \leq 3\}$, we can use the following expression of r_t:

$$r_t = \sum_{v \in P^*_t} \gamma^*_v \eta(v) + \sum_{w \in Q_t} \gamma_w \eta(w). \tag{12}$$

Lemma 4.4 *Let $n_t = |Q_t|$ be as described in* (9)*, and let M^*_t and N_t as above. Then $|N_t| > n_t$ for all $t \in \{1, \cdots, m\}$. In particular, $|M^*_t| > n_t$.*

Proof. Since $M^*_t \supseteq N_t$, it suffices to show that $|N_t| > n_t$. Suppose, to the contrary, that $|N_t| \leq n_t$ for some $t \in \{1, \cdots, m\}$. If $n_t = 1$ then $|N_t| = |M_t|$ by Remark 4.2 (i), which implies $|N_t| = 3m_t \geq 3 > 1 = n_t$, a contradiction. Hence we have $n_t > 1$.

Let $\mathcal{G} = (V, E)$ be the graph with the vertex set $V = M_t$ and the edge set E defined by

$$\{vw \mid v, w \in V,\ v \neq w,\ \eta(t, 1, v) = \eta(t, 1, w)\}.$$

It is obvious that $\mathcal{G}_W = (W, E)$ is a clique graph, where $W = V \setminus N_t$. We set $V_{jl} = \{(l, i, j) \in V \mid 1 \leq i \leq m_t\}$ and $V_j = \bigcup_{l=1}^{3} V_{jl}$; thus $V = \bigcup_{j=1}^{n_t} V_j$. Let $\mathcal{G}^*_{V_j} = (V_j, E^*_{V_j})$ ($j = 1, \cdots, n_t$) be the complete 3-partite graph with the partite set $\{V_{j1}, V_{j2}, V_{j3}\}$ and let $\mathcal{G}^* = (V, E^*) = \bigcup_{j=1}^{n_t} \mathcal{G}^*_{V_j}$. Since $U(v) = V_j \setminus V_{jl} \cup \{v\}$ for $v \in V_{jl}$ and $\mathcal{G}_W$ is a clique graph, by Remark 4.2 (i), $\mathcal{R} = (V, E, E^*)$ satisfies (R), and so $\mathcal{R}$ is an R-graph; in fact, it is a non-empty clique R-graph. In addition, since $\mathcal{G}^*_{V_j}$ is the complete 3-partite graph, $\mathcal{R}$ is an R-colouring R-graph with $\mathfrak{C}_{n_t}(V) = \{V_1, \cdots, V_{n_t}\}$. Since $m_t > 0$, we see then that $|V_j| = 3m_t \geq 2m_t + 1 = 2\mathbf{m}(V_j) + 1$ for each $j \in \{1, \cdots, n_t\}$. Moreover, according to our hypothesis, $|N_t| \leq n_t$, that is, $|I(V)| \leq n_t$. Hence, by virtue of Theorem 3.11, a clique R-graph $\mathcal{R}$ has an R-cycle consisting of edges. That is, there exist $p > 1$ and edges $e_1, \cdots, e_p \in E$ with $e_q = v_q w_q$ ($1 \leq q \leq p$) such that all of v_q's and w_q's are different from each other, $w_q v_{q+1} \in E^*$ ($1 \leq q \leq p-1$) and $w_p v_1 \in E^*$. Let $v_q = (l_q, i_q, j_q)$ and $w_q = (l'_q, i'_q, j'_q)$, where $1 \leq q \leq p$. Let $\xi_t(1, l_q, i_q) = x_{t1} z_{tl_q} f_{ti_q} z_{tl_q}$. Then $e_q = v_q w_q \in E$ implies

$$\xi_t(1, l_q, i_q) g_{tj_q} = \eta(t, 1, v_q) = \eta(t, 1, w_q) = \xi_t(1, l'_q, i'_q) g_{tj'_q}.$$

Moreover, $w_q v_{q+1} \in E^*$ and $w_p v_1 \in E^*$ mean that $j'_q = j_{q+1}$, $j'_p = j_1$, and $l'_q \neq l_{q+1}$. Hence we have

$$\prod_{q=1}^{p} \xi_t(1, l_q, i_q)^{-1} \xi_t(1, l'_q, i'_q) = 1 \quad \text{with} \quad l'_q \neq l_{q+1} \ (1 \leq q \leq p-1).$$

Since $v_q \neq w_q$ and $\eta(t, 1, v_q) = \eta(t, 1, w_q)$ for all $1 \leq q \leq p$, it follows from Remark 4.2 (ii) that $(l_q, i_q) \neq (l'_q, i'_q)$ for all $1 \leq q \leq p$. However, this contradicts the assertion of Remark 4.1 (ii). $\square$

We are now in a position to prove Theorem 1.1.

Proof. [Proof of Theorem 1.1] (1): Let ρ be as described in (7); a non-trivial right ideal of RG. By virtue of Proposition 2.1, it suffices to show that ρ is proper. Let $r = \sum_{t=1}^{m} r_t$ be as described in (11); a non-zero element of ρ. For $t \in \{1, \cdots, m\}$, recall $P^*_t = \{(t, k, \mu) \mid \mu \in M^*_t,\ 1 \leq k \leq 3\}$ and $Q_t = \{(t, j) \mid 1 \leq j \leq n_t\}$. We set $P^* = \cup_{t=1}^{m} P^*_t$ and $Q^* = (\cup_{t=1}^{m} Q_t) \cup \{w_0\}$, where $w_0 = (0, 0)$. We define $\gamma_{w_0} = -1$ ($\in R$) and $\eta(w_0) = 1$ ($\in G$) respectively. Then $r = \sum_{v \in P^*} \gamma^*_v \eta(v) + \sum_{w \in Q^*} \gamma_w \eta(w) + 1$ by (12). In order to prove that ρ is proper, it suffices to show that $r \neq 1$. Suppose, to the contrary, that $r - 1 = 0$, that is,

$$\sum_{v \in P^*} \gamma_v^* \eta(v) + \sum_{w \in Q^*} \gamma_w \eta(w) = 0. \tag{13}$$

Now, we set $V = P^* \cup Q^*$ and let $\mathcal{G} = (V, E)$ be the graph whose vertices are the elements of V and whose edge set E is defined as

$$E = \{vw \mid v, w \in V,\ v \neq w,\ \eta(v) = \eta(w) \text{ in } G\}.$$

By (13), $\mathcal{G}$ is proper and it is a non-empty clique graph. Let $E^* = \{vw \mid v = (t, k, \mu), w = (t, k', \mu) \in P^*, k \neq k'\}$. By Remark 4.3, $\mathcal{R} = (V, E, E^*)$ satisfies (R′), and so $\mathcal{R}$ is an R-graph, and in fact, it is a non-empty clique R-graph. We shall show that $\mathcal{R}$ has an R-cycle. If $v \in V$,

$$U(v) = \begin{cases} \{(t, k', l, i, j) \mid 1 \leq k' \leq 3\} \text{ if } v = (t, k, l, i, j) \in P^* \\ \{v\} \text{ if } v \in Q^* \end{cases},$$

and so $\mathfrak{U} = \{U(v) \mid v \in V\}$ satisfies the condition (SC). Hence, either $\mathcal{R}$ is R-simple or it has an R-cycle of length 2 consisting of edges. We may assume, therefore, that $\mathcal{R}$ is R-simple. If $v \in P^*$ then $|U(v)| = 3$, and if $v \in Q^*$ then $|U(v)| = 1$. This means that $\mathfrak{L}(\mathcal{R}) = \{U(v) \mid v \in P^*\}$ and $\mathfrak{N}(\mathcal{R}) = \{U(v) \mid v \in Q^*\}$. By Lemma 4.4, $|M_t^*| > n_t$, which implies

$$|\mathfrak{L}(\mathcal{R})| = (1/3)|P^*| = \textstyle\sum_{t=1}^m |M_t^*| > \sum_{t=1}^m n_t.$$

On the other hand,

$$|\mathfrak{N}(\mathcal{R})| = |Q^*| = \textstyle\sum_{t=1}^m |Q_t| + 1 = \sum_{t=1}^m n_t + 1,$$

and so $|\mathfrak{L}(\mathcal{R})| \geq |\mathfrak{N}(\mathcal{R})|$. Hence, by Theorem 3.16, a clique R-graph $\mathcal{R}$ has an R-cycle consisting of edges, as desired. By the definition of an R-cycle, there exist $p > 1$ and $v_q = (t_q, k_q, \mu_q), w_q = (t_q, h_q, \mu_q) \in P^*$ with $\mu_q = (l_q, i_q, j_q) \in M_{t_q}^*$ $(1 \leq q \leq p)$ such that all of v_q's and w_q's are different from each other, $w_q v_q \in E^*$, $e_q = v_q w_{q+1} \in E$ $(1 \leq q \leq p-1)$ and $v_p w_1 \in E$. Hence, we have

$$\eta(v_q) = \eta(w_{q+1})\ (1 \leq q \leq p-1), \quad \eta(v_p) = \eta(w_1) \tag{14}$$

$$\text{and } \ k_q \neq h_q\ (1 \leq q \leq p). \tag{15}$$

For the sake of simplicity of notation, the subscript $p+1$ means the subscript 1; we set $w_{p+1} = w_1$, $t_{p+1} = t_1$, $j_{p+1} = j_1 \cdots$. Since $v_q \neq w_{q+1}$, by Remark 4.2 (i), $\eta(v_q) = \eta(w_{q+1})$ implies

$$\text{either} \quad t_q \neq t_{q+1} \quad \text{or} \quad j_q \neq j_{q+1}. \tag{16}$$

By (10), the definition of $\eta(v_q)$,

$$\eta(v_q) = x_{t_q k_q} \zeta_q \quad \text{and} \quad \eta(w_{q+1}) = x_{t_{q+1} h_{q+1}} \zeta_{q+1},$$

where $\zeta_q = z_{t_q l_q} f_{t_q i_q} z_{t_q l_q} g_{t_q j_q}$, and so (14) implies that

$$\prod_{q=1}^{p} (x_{t_q k_q})^{-1} x_{t_{q+1} h_{q+1}} = 1. \tag{17}$$

Recall that x_{tk}'s are elements in X which is a basis of a free group and that $x_{tk} = x_{t'k'}$ if and only if $(t,k) = (t',k')$. Now, in (17), if $t_q \neq t_{q+1}$, it is obvious that $x_{t_q k_q} \neq x_{t_{q+1} h_{q+1}}$. If $t_q = t_{q+1}$ then $j_q \neq j_{q+1}$ because of (16), and so $\mu_q \neq \mu_{q+1}$. Since $\eta(v_q) = \eta(w_{q+1})$ by (14), Remark 4.3 implies $k_q \neq h_{q+1}$, and hence, $x_{t_q k_q} \neq x_{t_{q+1} h_{q+1}}$ again. Moreover, we have that $x_{t_{q+1} h_{q+1}} \neq x_{t_{q+1} k_{q+1}}$ by (15). Therefore it follows a contradiction that $\prod_{q=1}^{p} (x_{t_q k_q})^{-1} x_{t_{q+1} h_{q+1}} \neq 1$. This complete the proof of (1).

(2): If K' is the prime field of K then $|K'| \leq |G|$, and therefore $K'G$ is primitive by (1). Since $\Delta(G) = 1$ by Lemma 2.2 (1), the conclusion follows from Lemma 2.5. $\square$

Now, a ring R is called a (right) strongly prime ring if for each $0 \neq \alpha \in R$, there exists a finite subset $S(\alpha)$ of R such that $\alpha S(\alpha)\beta \neq 0$ for all non-zero $\beta \in R$. $S(\alpha)$ is called a (right) insulator of α. For instance, domains and simple rings are strongly prime. Formanek's result [7, Theorem] on primitivity of RG for a domain R was generalized to one for a strongly prime ring R by Lawrence [12]. The same situation holds for the case of our theorem.

Corollary 4.5 *The assertion of Theorem 1.1 (1) holds also for a strongly prime ring R.*

Proof. Let $\varphi(x) = \sum_{i=1}^{m_x} \alpha_{xi} f_{xi}$ $(x \in X)$ be as described in (5) and $S(\alpha_{x1}) = \{\delta_{xq} \mid 1 \leq q \leq d_x\}$ a right insulator of α_{x1}. Going back to the beginning of this section, for $\{x^{(1)}, x^{(2)}, x^{(3)}, f_{xi} \mid 1 \leq i \leq m_x\}$, there exist elements z_{xlq} $(1 \leq l \leq 3, 1 \leq q \leq d_x)$

which satisfy assertion of Lemma 2.2 (2). We replace (6) by

$$\begin{aligned}\varepsilon(x) &= \sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{q=1}^{d_x} x^{(k)} z_{xlq}\psi(x)\delta_{xq} z_{xlq} + 1\\ &= \sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{q=1}^{d_x}\sum_{i=1}^{m_x} \alpha_{xi}\delta_{xq} x^{(k)} z_{xlq} f_{xi} z_{xlq} + 1.\end{aligned}$$

Then (8) is replaced by

$$\begin{aligned}r_t = &\sum_{k,l=1}^{3}\sum_{q=1}^{d_t}\sum_{i=1}^{m_t}\sum_{j=1}^{n_t} \alpha_{ti}\delta_{tq}\beta_{tj} x_{tk} z_{tlq} f_{ti} z_{tlq} g_{tj}\\ &\qquad + \sum_{j=1}^{n_t}\beta_{tj} g_{tj}, \text{ where } m_t, n_t, d_t > 0.\end{aligned} \tag{18}$$

Let $A_{tj} = \{q \mid 1 \leq q \leq d_x, \alpha_{ti}\delta_{tq}\beta_{tj} \neq 0 \text{ for some } i\}$. Since $S(\alpha_{t1})$ is a right insulator of α_{t1}, for each $j \in \{1, \cdots, n_t\}$, there exists $q \in \{1, \cdots, d_t\}$ such that $\alpha_{t1}\delta_{tq}\beta_{tj} \neq 0$, and so $A_{tj} \neq \emptyset$. For $q \in A_{tj}$, let $B_{tj}(q) = \{i \mid 1 \leq i \leq m_t, \alpha_{ti}\delta_{tq}\beta_{tj} \neq 0\}$. It is obvious that $B_{tj}(q) \neq \emptyset$. The non-zero parts of (18) is here replaced by the following expression:

$$\begin{aligned}r_t = &\sum_{k,l=1}^{3}\sum_{j=1}^{n_t}\sum_{q\in A_{tj}}\sum_{i\in B_{tj}(q)} \alpha_{ti}\delta_{tq}\beta_{tj} x_{tk} z_{tlq} f_{ti} z_{tlq} g_{tj}\\ &+ \sum_{j=1}^{n_t}\beta_{tj} g_{tj}, \text{ where } n_t > 0, |A_{tj}| > 0, |B_{tj}(q)| > 0.\end{aligned} \tag{19}$$

After this, we renumber the elements in A_{tj} and $B_{tj}(q)$, and we can then follow the same proof as in Theorem 1.1 (1).

We can summarize the procedure as follows: Let $A_{tj} = \{q_1, \cdots, q_{a_{tj}}\}$; $a_{tj} = |A_{tj}|$, and for $q_s \in A_{tj}$, let $B_{tj}(q_s) = \{p_1, \cdots, p_{m_{tjs}}\}$; $m_{tjs} = |B_{tj}(q_s)|$. We set $A^*_{tj} = \{1, \cdots, 3a_{tj}\}$ and $B^*_{tjl} = \{1, \cdots, m_{tjs}\}$, where for $1 \leq h \leq 3$, $l = 3(s-1) + h$. We here replace P_t in (9) by

$$P_t = \{(t, k, l, i, j) \mid 1 \leq k \leq 3,\ l \in A^*_{tj},\ i \in B^*_{tjl},\ 1 \leq j \leq n_t\}.$$

Then $\eta(t, k, l, i, j) = x_{tk} z_{thq_s} f_{tp_i} z_{thq_s} g_{tj}$, where $l = 3(s-1) + h$ with $1 \leq h \leq 3$ and $p_i \in B_{tj}(q_s)$. We also replace M_t by $M_t = \{(l, i, j) \mid l \in A^*_{tj},\ i \in B^*_{tjl},\ 1 \leq j \leq n_t\}$. Let $\mathcal{R} = (V, E, E^*)$ be as described in the proof of Lemma 4.4, where $V = M_t$ as above. Then $\mathcal{R}$ is an R-colouring R-graph with $\mathfrak{C}_{n_t} = \{V_1, \cdots, V_{n_t}\}$, and the difference between this $\mathcal{R}$ and the one in the proof of Lemma 4.4 is simply that $V_j = \bigcup_{l=1}^{3} V_{jl}$ and $|V_{jl}| = m_t$ there whereas $V_j = \bigcup_{l=1}^{3a_{tj}} V_{jl}$ and $|V_{jl}| = m_{tjs}$ here. Since $a_{tj} > 0$ and $m_{tjs} > 0$, we can easily see that Theorem 3.11 is also valid in this case and that the same assertion as Lemma 4.4 holds. The remains of the proof are the same as the proof of Theorem 1.1 (1). $\square$

If $F_1 \subseteq F_2 \subseteq \cdots$ are free groups, then $F_\infty = \bigcup_{i=1}^{\infty} F_i$ contains a free subgroup F with $|F| = |F_\infty|$. In fact, if either $|F_i| \leq \aleph_0$ for all i or $|F_i|$ is a maximal cardinality for some i, then the assertion is obvious. Hence, it suffices to consider the case that their cardinalities are not bounded above. We may then assume that $|F_i| < |F_{i+1}|$ for all i. Since each element of F_i is a product of finitely many basis elements of F_{i+1}, each F_{i+1} can be written as a free product $G_{i+1} * H_{i+1}$, where G_{i+1} and H_{i+1} are free subgroups of F_{i+1} with $F_i \subseteq G_{i+1}$ and $|F_{i+1}| = |H_{i+1}|$. Then $H_2 * H_3 * \cdots$ is a free subgroup of F_∞ with the same cardinality as F_∞. Now, it is well known that a countable locally free group is the union of an ascending sequence of free subgroups. Hence, by Theorem 1.1 (2), we have

Corollary 4.6 *Let $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$ be an ascending chain of non-abelian free groups, and $F_\infty = \cup_{i=1}^{\infty} F_i$. Then the group ring KF_∞ is primitive for any field K. In particular, every group ring of a countable non-abelian locally free group over a field is primitive.*

We are now in a position to prove easily Theorem 1.2:

Proof. [Proof of Theorem 1.2] By virtue of Lemma 2.6, we may assume that $\varphi(F) \neq F$. Then $\Delta(F_\varphi) = 1$ by lemma 2.3 (2). Let F_i be the subgroup of F_φ generated by $\{t^i f t^{-i} \mid f \in F\}$, and $F_\infty = \bigcup_{i=1}^{\infty} F_i$. By lemma 2.3 (3), F_∞ is a normal subgroup of F_φ, and it is also a locally free group which is of type as described in Corollary 4.6. Hence, KF_∞ is primitive by Corollary 4.6. It is obvious that F_φ / F_∞ is isomorphic to $\langle t \rangle$, and thereby, it follows from Lemma 2.4 (1) that KF_φ is primitive. □

Finally, we state the semiprimitivity of group rings of ascending HNN extensions of free groups, which extends [16, Corollary 3.7] to the general cardinality case:

Corollary 4.7 *Let F be a non-abelian free group, and F_φ the ascending HNN extension of F determined by φ. If K is any field then the group ring KF_φ is semiprimitive.*

Proof. Let K' be the prime field of K. Since $|K'| \leq |F|$, by virtue of Theorem 1.2, $K'F_\varphi$ is primitive and so semiprimitive. As is well known, semiprimitive group rings are separable algebras, thus semiprimitivity of group rings close under extensions of coefficient fields, and therefore KF_φ is semiprimitive. □

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